TOPOLOGY-Midsem test, BMath-II

INSTRUCTIONS: Total time 3 hours. Solve as many problems as you like, for a maximum score of 30 marks. Please use notations and terminology as developed in the course, use results done in the class without proving them. If you use a problem from some assignment/homework/book, please provide its solution.

1. Let X be a metric/topological space and $Y \subset X$ be dense. Let $f, g : X \to Z$ be continuous functions, where Z is another metric/topological space. Prove that if f and g satisfy $f(y) = g(y)$ for all $y \in Y$, then $f = g$. (4)

2. Let X, Y, Z be metric spaces. Show by an example that a function $f : X \times Y \to Z$ may be continuous in each variable, yet fail to be continuous. (6)

3. Let $f : X \to Y$ be a function. The graph of f is defined by $\Gamma_f = \{(x, f(x)),\ x \in X\}$. Assume X and Y are metric/topological spaces. Prove that f is continuous implies Γ_f is a closed subset of $X \times Y$. Is the converse true? Justify your answer. (4+4)

4. Let X be a topological space and $\mathcal{C}(X, \mathbb{R})$ denote the metric space of all bounded (real valued) continuous functions on X. Assume that $\mathcal{C}(X, \mathbb{R})$ separates points in X, i.e., given $x \neq y$ in X, there is $f \in \mathcal{C}(X, \mathbb{R})$ such that $f(x) \neq f(y)$. Prove that X must be Hausdorff. (6)

5. Let X be a compact Hausdorff topological space. Let $x \in X$ and $C \subset X$ be a closed set not containing x. Construct open subsets U, V of X such that $x \in U,\ C \subset V$ and $U \cap V = \emptyset$. (6)

6. Determine if the metric space $\mathcal{C}([0, 1], \mathbb{R})$ is second countable, i.e., admits a countable basis of open sets. (8)

7. Let Y be a subspace of a topological/metric space X and $A \subset Y$. Denote the boundary of A in Y by $\partial_Y A$ and the boundary of A in X by ∂A. Is it then true that $\partial_Y A = (\partial A) \cap Y$? Explain your answer. (6)